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New Inhomogeneous Einstein Metrics on Sphere Bundles Over Einstein-Kähler Manifolds

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ABSTRACT

We construct new complete, compact, inhomogeneous Einstein metrics on S^{m+2} sphere bundles over $2n$ -dimensional Einstein-Kähler spaces K_{2n} , for all $n \geq 1$ and all $m \geq 1$. We also obtain complete, compact, inhomogeneous Einstein metrics on warped products of S^m with S^2 bundles over K_{2n} , for $m > 1$. Additionally, we construct new complete, non-compact Ricci-flat metrics with topologies S^m times \mathbb{R}^2 bundles over K_{2n} that generalise the higher-dimensional Taub-BOLT metrics, and with topologies $S^m \times \mathbb{R}^{2n+2}$ that generalise the higher-dimensional Taub-NUT metrics, again for $m > 1$.

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1 Introduction

Compact homogeneous Einstein spaces, in the form of cosets G/H , are well known. Less well known are the examples of complete, compact Einstein spaces that are inhomogeneous. The first explicit example was obtained in [1], by considering a limit of the four-dimensional Euclidean Kerr-de Sitter solution. It has the topology of the non-trivial S^2 bundle over S^2 , which is $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. A wide class of complete, compact inhomogeneous Einstein metrics in arbitrary even dimensions $d = 2n + 2$ was then obtained [2, 3]. These are defined on manifolds which are S^2 bundles over K_{2n} , where K_{2n} is an Einstein-Kähler space. The original example in [1] is the special case where $n = 1$, with K_2 then being the 2-sphere with its standard Einstein-Kähler metric.

Further explicit examples of complete, compact, inhomogeneous Einstein spaces have recently been obtained [4]. Like the original example in [1], these were again found by considering limits of Kerr-de Sitter black holes, this time in $d \geq 5$ dimensions. Specifically, in $d = 5$ the general Kerr-de Sitter black hole with two independent rotation parameters is known [5]. By taking limiting cases of the Euclideanised black-hole solutions, it was shown in [4] that these metrics encompass a countable infinity of inequivalent complete, compact Einstein metrics. The topologies of the associated smooth manifolds are either $S^3 \times S^2$ or the non-trivial S^3 bundle over S^2 , according to whether a certain integer is even or odd [4]. In addition, further complete, compact inhomogeneous metrics were constructed in all dimensions $d > 5$ in [4]. These metrics, one for each dimension $d > 5$, were obtained by taking a limit of the Euclideanised d -dimensional Kerr-de Sitter black hole with one rotation parameter.¹ The topology of the manifold is the non-trivial S^{d-2} bundle over S^2 [4].

In this paper, we shall give a construction of complete, compact, inhomogeneous Einstein metrics on S^{m+2} bundles over Einstein-Kähler spaces K_{2n} of any (even) dimension $2n \geq 2$. The metrics on the S^{d-2} bundles over S^2 obtained in [4] are thus special cases of our new metrics, when the Einstein-Kähler base is taken to be S^2 . Our construction is also a natural generalisation of the one in [2, 3].

We also obtain a new infinite class of complete, compact, inhomogeneous Einstein metrics on warped products of S^m with S^2 bundles over K_{2n} , for $m > 1$. These are different from the direct-product Einstein metrics that one could trivially construct from the previous results in [2, 3].

In addition to the new compact Einstein metrics described above, we also obtain new

¹Examples of new, inhomogeneous Einstein-Sasaki metrics in five [6] and higher [7] dimensions have been obtained recently too.

complete, non-compact Ricci-flat metrics that generalise the Taub-NUT and Taub-BOLT metrics in the literature. These have the topologies $S^m \times \mathbb{R}^{n+2}$, and S^m times \mathbb{R}^2 bundles over K_{2n} respectively, with $m > 1$.

2 The Local Solutions

Let $d\Sigma_{2n}^2$ be an Einstein-Kähler metric of real dimension $2n$, normalised such that

$$R_{ab} = \lambda g_{ab}. \quad (1)$$

In this paper, we shall be concentrating on the case where $\lambda > 0$. We then write the following ansatz for metrics of dimension $d = 2n + m + 2$:

$$d\hat{s}^2 = dt^2 + \beta^2 (d\tau - 2A)^2 + \gamma^2 d\Sigma_{2n}^2 + \delta^2 d\Omega_m^2, \quad (2)$$

where β , γ and δ are functions of t , $d\Omega_m^2$ is the standard metric on the unit sphere S^m , and A is a potential for the Kähler form on $d\Sigma_{2n}^2$: $J = dA$. This metric is similar to the one considered in [3], except that now an extra m dimensions involving the m -sphere are included. A straightforward calculation shows that in the orthonormal frame $\hat{e}^0 = dt$, $\hat{e}^1 = \beta (d\tau - 2A)$, $\hat{e}^a = \gamma e^a$, $\hat{e}^i = \delta e^i$, the non-vanishing components of the Ricci tensor are given by

$$\begin{aligned} \hat{R}_{00} &= -\frac{\ddot{\beta}}{\beta} - 2n \frac{\ddot{\gamma}}{\gamma} - m \frac{\ddot{\delta}}{\delta}, \\ \hat{R}_{11} &= -\frac{\ddot{\beta}}{\beta} - 2n \frac{\dot{\beta}\dot{\gamma}}{\beta\gamma} - m \frac{\dot{\beta}\dot{\delta}}{\beta\delta} + 2n \frac{\beta^2}{\gamma^4}, \\ \hat{R}_{ab} &= -\left(\frac{\ddot{\gamma}}{\gamma} + \frac{\dot{\beta}\dot{\gamma}}{\beta\gamma} + m \frac{\dot{\gamma}\dot{\delta}}{\gamma\delta} + (2n-1) \frac{\dot{\gamma}^2}{\gamma^2} + \frac{2\beta^2}{\gamma^4} - \frac{\lambda}{\gamma^2}\right) \delta_{ab}, \\ \hat{R}_{ij} &= -\left(\frac{\ddot{\delta}}{\delta} + \frac{\dot{\beta}\dot{\delta}}{\beta\delta} + 2n \frac{\dot{\gamma}\dot{\delta}}{\gamma\delta} + (m-1) \frac{\dot{\delta}^2}{\delta^2} - \frac{m-1}{\delta^2}\right) \delta_{ij}. \end{aligned} \quad (3)$$

One can look for solutions to the Einstein equation

$$\hat{R}_{AB} = \Lambda \delta_{AB} \quad (4)$$

by following a procedure similar to the one described in [3]. We find that a solution is given by

$$d\hat{s}^2 = \frac{(1-r^2)^n dr^2}{P(r)} + \frac{c^2 P(r)}{(1-r^2)^n} (d\tau - 2A)^2 + c(1-r^2) d\Sigma_{2n}^2 + \frac{(m-1)}{\Lambda - \lambda c^{-1}} r^2 d\Omega_m^2, \quad (5)$$

where c is an arbitrary constant of integration and the function $P(r)$ is given by

$$\frac{d(r^{m-1} P(r))}{dr} = r^{m-2} [\Lambda (1-r^2)^{n+1} - \lambda c^{-1} (1-r^2)^n]. \quad (6)$$

Thus we have

$$P(r) = \frac{\Lambda}{m-1} {}_2F_1(-n-1, \frac{m-1}{2}; \frac{m+1}{2}; r^2) - \frac{\lambda c^{-1}}{m-1} {}_2F_1(-n, \frac{m-1}{2}; \frac{m+1}{2}; r^2) + \mu r^{1-m}, \quad (7)$$

where ${}_2F_1$ denotes the standard hypergeometric function. The quantity μ in the final term is a further constant of integration. The rest of the terms in $P(r)$ form a polynomial in r of degree $2n$. It is useful to note that the hypergeometric functions that arise here have the form

$$\begin{aligned} \frac{1}{m-1} {}_2F_1(-1, \frac{m-1}{2}; \frac{m+1}{2}; r^2) &= \frac{1}{m-1} - \frac{r^2}{m+1}, \\ \frac{1}{m-1} {}_2F_1(-2, \frac{m-1}{2}; \frac{m+1}{2}; r^2) &= \frac{1}{m-1} - \frac{2r^2}{m+1} + \frac{r^4}{m+3}, \\ \frac{1}{m-1} {}_2F_1(-3, \frac{m-1}{2}; \frac{m+1}{2}; r^2) &= \frac{1}{m-1} - \frac{3r^2}{m+1} + \frac{3r^4}{m+3} - \frac{r^6}{m+5}, \end{aligned} \quad (8)$$

etc., with the pattern continuing to higher values of n in the obvious way.

There are in total two non-trivial parameters in the Einstein metrics (5). One is the integration constant μ , and the other can be taken to be the dimensionless constant

$$\nu \equiv \frac{c\Lambda}{\lambda}. \quad (9)$$

The constant Λ is just the overall scale-setting Einstein constant in (4), and it can easily be seen by rescaling the Einstein-Kähler base metric that λ and c always occur in the combination λc^{-1} . We shall make various convenient choices for the “trivial” parameters Λ and λ as the occasion arises.

The case $m = 1$ is degenerate in the above parameterisation. However, it can easily be obtained as a limiting case, in which one absorbs the $(m-1)$ factor into a rescaling of $d\Omega_m^2$ before setting $m = 1$, and so we have, with $0 \leq \chi \leq 2\pi$ forming an S^1 ,

$$d\hat{s}^2 = \frac{(1-r^2)^n dr^2}{P(r)} + \frac{c^2 P(r)}{(1-r^2)^n} (d\tau - 2A)^2 + c(1-r^2) d\Sigma_{2n}^2 + P_0^{-1} r^2 d\chi^2, \quad (10)$$

where P_0 is an arbitrary positive constant. The Einstein equations now require $\nu = 1$ and that $P(r)$ satisfies $P'(r) = -\Lambda r (1-r^2)^n$ and hence $P(r)$ is given by

$$P(r) = \frac{\Lambda}{2(n+1)} [(1-r^2)^{n+1} - 1] + \mu. \quad (11)$$

where μ is an integration constant. The metric for $m = 1$ thus has two non-trivial parameters, namely μ and P_0 .

In certain low-dimensional cases, namely $n = 1$ with $m = 1, 2$ and 3 , and $n = 2$ with $m = 1$, some local solutions that overlap with ours were obtained in [8].

The local Einstein metrics we have obtained are in general singular, in the sense that they cannot extend onto smooth manifolds. For special choices of the parameters, however, non-singular metrics arise. The issue of whether this occurs or not can be understood by studying the local forms of the metrics at the “endpoints” of the range of the radial coordinate r . The idea is that r ranges between two adjacent values, say r_1 and r_2 , at which one or more of the metric functions β , γ or δ in (2) vanishes. At such points, one can obtain regular behaviour if it should happen that the vanishing metric functions imply the collapse of spheres to an origin of spherical polar coordinates. If the collapsing sphere has dimension greater than 1, the Einstein equations imply that the rate of collapse will automatically be that needed for regularity. In the case of a one-dimensional sphere, i.e. a circle, S^1 , its rate of collapse is not governed by the Einstein equations, and it places a condition on the periodicity of the S^1 coordinate in order to avoid a conical singularity.

3 Complete Metrics on S^{m+2} Bundles Over K_{2n}

In this section, we study the circumstances under which the above local solutions extend smoothly onto manifolds that are S^{m+2} bundles over the K_{2n} Einstein-Kähler base spaces. In this case one endpoint for the radial coordinate is $r = r_1 = 0$, signalling a collapse of the sphere S^m , whose metric is $d\Omega_m^2$ in (5) or $d\chi^2$ in (10). Without loss of generality we can take $r \geq 0$, so that the upper endpoint will occur at $r = r_2$, the first positive root of the function $P(r)$, where $\beta = 0$:

$$P(r_2) = 0. \tag{12}$$

The cases $m = 1$ and $m > 1$ require separate discussions, and we shall consider the latter first.

3.1 $m > 1$: S^{m+2} bundles over K_{2n}

In order to have regular solutions, the function $P(r)$ must be regular at $r = 0$, which for $m > 1$ implies that the integration constant μ in (7) must vanish, and hence $P(r)$ is a polynomial in r of degree $2n$. As discussed above, the Einstein equations automatically imply in this case that the spheres S^m collapse in a regular fashion at $r = 0$. The question of regularity thus devolves upon the behaviour at the first positive zero r_2 of the polynomial $P(r)$, at which the length of the $U(1)$ fibres parameterised by τ goes to zero.

Before considering the regularity conditions at $r = r_2$, it should be noted that the possible periods for the $U(1)$ fibre coordinate τ are dictated by the first Chern class c_1 of the tangent

bundle of the Einstein-Kähler space K_{2n} . This was discussed in detail in [3], and here we shall just summarise the conclusion. The allowed periods for τ are given by

$$\Delta\tau = \frac{4\pi p}{k\lambda}, \quad (13)$$

where k is any positive integer, and p is a non-negative integer, defined as that integer such that c_1 evaluated on $H_2(K_{2n}, \mathbb{Z})$ is $\mathbb{Z} \cdot p$, i.e. the integers divisible by p . Amongst all the Einstein-Kähler manifolds K_{2n} , the integer p is maximised in \mathbb{CP}^n , for which $p = n + 1$.

In order for the metric (5) to be regular at $r = r_2$, it must be the case that the required periodicity for τ is contained within the set of allowed values given in (13). By writing $P(r) \sim -P'(r_2)(r_2 - r) = -P'(r_2)\rho^2$ near $r = r_2$, one finds that the metric (5) restricted to the (ρ, τ) plane has no conical singularity if τ has period given by

$$\Delta\tau = \frac{4\pi(1 - r_2^2)^n}{c(-P'(r_2))}, \quad (14)$$

where $P'(r)$ is the derivative of $P(r)$ with respect to r , which is negative at $r = r_2$. Combining this with (13), and making use of (6), we therefore obtain the condition

$$\frac{k}{p} = \frac{1 - \nu(1 - r_2^2)}{r_2}. \quad (15)$$

Note that now $\nu \equiv c\Lambda/\lambda$ is the single remaining non-trivial parameter in the Einstein metrics (5), since we have already set $\mu = 0$ for regularity at $r = 0$. The solution (7) for $P(r)$, with $\mu = 0$, implies that $P(r_2) = 0$ gives

$$\nu = \frac{{}_2F_1(-n, \frac{m-1}{2}; \frac{m+1}{2}; r_2^2)}{{}_2F_1(-n-1, \frac{m-1}{2}; \frac{m+1}{2}; r_2^2)}, \quad (16)$$

and the regularity condition (15) then gives

$$\frac{k}{p} = \kappa(r_2) \equiv \frac{1}{r_2} \left[1 - (1 - r_2)^2 \frac{{}_2F_1(-n, \frac{m-1}{2}; \frac{m+1}{2}; r_2^2)}{{}_2F_1(-n-1, \frac{m-1}{2}; \frac{m+1}{2}; r_2^2)} \right], \quad (17)$$

which is the one non-trivial condition (determining r_2) for a regular metric.

For small r_2 we find that $\kappa(r_2) \sim 2r_2/(m+1)$, and it is obvious from (17) that $\kappa(1) = 1$. It follows from the continuity of $\kappa(r)$ that for any integer k in the range

$$0 < k < p, \quad (18)$$

we can find a corresponding r_2 with $0 < r_2 < 1$ that satisfies (17). Equation (16) then determines the value of the non-trivial parameter ν . Thus we have complete, nonsingular Einstein metrics for all integer k within the range (18).

Over each point in the Einstein-Kähler base space K_{2n} , the metric (5) describes a sphere S^{m+2} , foliated by $S^1 \times S^m$ surfaces at each value of r with $0 < r < r_2$. The S^m degenerates at $r = r_1 = 0$, and the $S^1 \sim U(1)$ degenerates at $r = r_2$. Since the $U(1)$ is fibred over the K_{2n} base, it follows that the total space is an S^{m+2} bundle over K_{2n} .

The special cases where $n = 1$, for which the Einstein-Kähler base space K_2 is just S^2 , reproduce the Einstein metrics on S^{d-2} bundles over S^2 that were obtained in [4]. Since $p = 2$ in this case, it follows from (18) that the integer k can only take the value $k = 1$. In this case, the function P is given by

$$P = \Lambda \left[\frac{1}{m-1} - \frac{2r^2}{m+1} + \frac{r^4}{m+3} - \frac{1-r_2^2}{1-k r_2/p} \left(\frac{1}{m-1} - \frac{r^2}{m+1} \right) \right], \quad (19)$$

with the constant r_2 satisfying the regularity condition

$$\frac{1}{m-1} - \frac{2r_2^2}{m+1} + \frac{r_2^4}{m+3} = 4r_2 \left[\frac{1}{(m-1)(m+1)} - \frac{r_2^2}{(m+1)(m+3)} \right]. \quad (20)$$

For $n = 2$, we have that $p = 3$, and hence $k = 1$ or $k = 2$. The function P is given by

$$P = \Lambda \left[\frac{1}{m-1} - \frac{3r^2}{m+1} + \frac{3r^4}{m+3} - \frac{r^6}{m+5} - \frac{1-r_2^2}{1-k r_2/p} \left(\frac{1}{m-1} - \frac{2r_2^2}{m+1} + \frac{r^4}{m+3} \right) \right], \quad (21)$$

where the constant r_2 satisfies the regularity condition

$$\begin{aligned} \frac{1}{m-1} - \frac{3r_2^2}{m+1} + \frac{3r_2^4}{m+3} - \frac{r_2^6}{m+5} = \\ \frac{2p}{k} \left[\frac{1}{(m-1)(m+1)} - \frac{2r_2^2}{(m+1)(m+3)} + \frac{r_2^4}{(m+3)(m+5)} \right]. \end{aligned} \quad (22)$$

As an explicit example, let us consider $m = 2$. We have $r_2 \approx 0.445844$ when $k = 1$, and $r_2 \approx 0.750285$ when $k = 2$.

3.2 $m = 1$: S^3 bundles over K_{2n}

We now consider the case where $m = 1$, which was not covered in section 3.1. The function $P(r)$ is then given by (11). The regularity condition (15) then implies that

$$\frac{k}{p} = r_2. \quad (23)$$

Thus we can have any k in the range $0 < k < p$, as in (18), with the root r_2 lying in the interval $0 < r_2 < 1$. Setting $P(r_2) = 0$ in (11) determines μ , and hence gives

$$P(r) = (1 - r^2)^{n+1} - (1 - k^2/p^2)^{n+1}, \quad (24)$$

where for simplicity, we have now made the conventional scale choice $\Lambda = 2n + 2$. With our choice that the coordinate χ on S^1 in the metric (10) has period 2π , the regularity

condition for no conical singularity at $r = 0$ implies that the constant P_0 in (10) is given by $P_0 = P(0)$.

In summary, we have complete, compact, inhomogeneous Einstein metrics for all k in the range (18), on spaces whose topologies are S^3 bundles over the Einstein-Kähler base space K_{2n} . Interestingly, these metrics are all fully explicit, with

$$d\hat{s}_{2n+3}^2 = \frac{(1-r^2)^n dr^2}{P(r)} + \frac{P(r)}{(1-r^2)^n} (d\tau - 2A)^2 + (1-r^2) d\Sigma_{2n}^2 + P_0^{-1} r^2 d\chi^2, \quad (25)$$

where $P(r)$ is given by (24), and here we have normalised the metric $d\Sigma_{2n}^2$ on the Einstein-Kähler base space K_{2n} so that $\lambda = 2n + 2$. The special case with $n = 1$, for which $K_2 = S^2$, was found in [4].

4 Warped-Product Einstein Metrics

In this section we consider a different class of complete Einstein metrics encompassed within the solutions (5) and (10), in which the radial coordinate r ranges between two adjacent positive zeroes of $P(r)$, at r_1 and r_2 . In order also to avoid the singularity of the metric components at $r = 1$, we require

$$0 < r_1 < r_2 < 1. \quad (26)$$

It is evident from the solution for $P(r)$ given in (11), which is monotonically decreasing as r increases from 0 to 1, that we cannot achieve two roots as in (26) when $m = 1$. Thus to obtain warped-product Einstein metrics we must restrict attention to the cases with $m > 1$.

Since $P(r)$ now vanishes at both endpoints, r_1 and r_2 , of the range of the radial coordinate, there is a regularity condition of the form (15) at each endpoint. Specifically, we now have

$$\frac{\nu(1-r_1^2)-1}{r_1} = \frac{k}{p} = \frac{1-\nu(1-r_2^2)}{r_2}, \quad (27)$$

where the reversal of sign in the condition at r_1 results from the fact that $P'(r_1)$ is positive, whereas $P'(r_2)$ is negative. Thus we have

$$\nu = \frac{1}{1-r_1 r_2}, \quad (28)$$

$$\frac{k}{p} = \frac{r_2 - r_1}{1 - r_1 r_2}. \quad (29)$$

It follows immediately from (29) that $k/p > 0$ and $1 - k/p > 0$, and hence k must lie in the range $0 < k < p$, as in (18).

By definition, the endpoints r_1 and r_2 satisfy $P(r_i) = 0$. We can easily arrange that $P(r_i) = 0$ for one endpoint by choosing the non-trivial parameter μ appropriately in (7).

For the $P(r_i) = 0$ at the other endpoint, we see that the integral of the right-hand side of (6) between the endpoints must vanish, which implies

$$D(r_1, r_2) \equiv \int_{r_1}^{r_2} dr r^{m-2} (1 - r^2)^n (r_1 r_2 - r^2) = 0. \quad (30)$$

For a given k/p , (29) and (30) give two algebraic equations for the unknowns r_1 and r_2 . We shall show that for each k in the range $0 < k < p$, there always exists a solution for r_1 and r_2 satisfying (26), and hence we have a complete, nonsingular Einstein space for each such k .

To show the existence of solutions to (29) and (30), it is helpful for now to view $R \equiv k/p$ as a continuous variable that satisfies

$$0 \leq R \leq 1. \quad (31)$$

Of course it is actually discrete, and the limits $R = 0$ and $R = 1$ are disallowed by (18), but it is easier first to discuss the existence of solutions to

$$R = \frac{r_2 - r_1}{1 - r_1 r_2}, \quad D(r_1, r_2) = 0, \quad (32)$$

for $0 \leq r_1 \leq r_2 \leq 1$.

We first consider the function $D(r_1, r_2)$ with $r_2 = 1$, which, from (32), implies $R = 1$. Clearly, $D(r_1, 1)$ is negative if $r_1 = 0$, whilst for r_1 close to 1 an expansion for $r_1 = 1 - \epsilon$ shows that $D(r_1, 1)$ is positive. By continuity, there therefore exists some r_1 for which $D(r_1, 1) = 0$. If r_2 is reduced from $r_2 = 1$, then R reduces too. $D(r_1, r_2)$ is again negative if $r_1 = 0$, whilst it is positive if r_1 is infinitesimally below r_2 , provided that

$$r_2^2 > \frac{m}{m + 2n}. \quad (33)$$

By continuity, there is therefore always a solution of $D(r_1, r_2) = 0$ provided that (33) is satisfied. In fact, if r_2^2 approaches $m/(m + 2n)$, then the solution to $D(r_1, r_2) = 0$ gives r_1 approaching r_2 , and hence, from (32), R approaches 0. Thus for any value of R with $0 < R < 1$ we have a solution for r_1 and r_2 with $0 < r_1 < r_2 < 1$.

We have therefore established that for all positive integers m , we have a complete, compact, nonsingular Einstein metric for each integer k with $0 < k < p$.

Since the coefficient of $d\Omega_m^2$ in (5) is non-vanishing for the entire range of the radial coordinate, $r_1 \leq r \leq r_2$, it follows that topologically the manifolds for these metrics are simply the product of S^m with the same class of S^2 bundles over K_{2n} that was discussed in [3]. Metrically, however, the Einstein spaces we have obtained here are *warped* products of S^m with the S^2 bundles over K_{2n} , since the size of the S^m varies with r .

Since the size of S^m is non-zero over the complete coordinate range $r_1 \leq r \leq r_2$, we do not need to restrict $d\Omega_m^2$ to be the Einstein metric on the unit S^m . We can take any Einstein space with $R_{ij} = (m-1)\delta_{ij}$ in place of S^m . This, of course, is different from the situation in section 3, where the coefficient of $d\Omega_m^2$ vanished at the lower endpoint, and so it necessarily had to be a round m -sphere for regularity at $r = 0$.

It should be emphasised that the Einstein spaces we have obtained in this section are distinct from examples with identical topology that we could trivially obtain by taking the direct product of the S^2 bundles over K_{2n} in [3] with appropriately-scaled metrics on S^m , or on any other Einstein space with positive Ricci tensor in place of S^m . Thus for every direct-product metric of this type, our new construction in this section yields an inequivalent warped-product Einstein space with the same topology.

Now let us present some explicit examples. First, consider the cases with $n = 1$ and $m \geq 2$. The function P is given by

$$P = \Lambda \left[\frac{r^4}{m+3} - \frac{(r_1 r_2 + 1)r^2}{m+1} + \frac{r_1 r_2}{m-1} - \left(\frac{r_1}{r}\right)^{m-1} \left(\frac{r_1^4}{m+3} - \frac{(r_1 r_2 + 1)r_1^2}{m+1} + \frac{r_1 r_2}{m-1} \right) \right]. \quad (34)$$

For a specific example consider the six-dimensional metric obtained by taking $m = 2$. The regularity conditions (30) and (29) with $k = 1$, $p = 1$ imply that

$$\begin{aligned} r_1 &= \frac{\sqrt{4\sqrt{85}-19} + \sqrt{85}-10}{6} \approx 0.574634, \\ r_2 &= \frac{\sqrt{4\sqrt{85}-19} - \sqrt{85}+10}{6} \approx 0.834786. \end{aligned} \quad (35)$$

Thus we have $\nu = 1 + \sqrt{17/20} \approx 1.92195$. For the seven-dimensional example with $m = 3$, we have

$$\begin{aligned} r_1 &= \frac{\sqrt{6\sqrt{21}-18} + \sqrt{21}-5}{4} \approx 0.666011, \\ r_2 &= \frac{\sqrt{6\sqrt{21}-18} - \sqrt{21}+5}{4} \approx 0.874723, \end{aligned} \quad (36)$$

and $\nu = \frac{1}{4}(5 + \sqrt{21}) \approx 2.39564$.

5 Non-Compact Ricci-Flat Metrics

Here, we show that one can obtain complete, nonsingular, non-compact Ricci-flat metrics from the solutions. Setting $\Lambda = 0$, $c = -1$ and $P(r) = (-1)^n \tilde{P}(r)$ in (2) and (10), we

find that when $m = 1$ there are no non-singular solutions within the ansatz that we are considering, and thus we shall concentrate on the cases with $m > 1$. For these, the metric (5) becomes

$$d\hat{s}^2 = \frac{(r^2 - 1)^n dr^2}{\tilde{P}(r)} + \frac{\tilde{P}(r)}{(r^2 - 1)^n} (d\tau - 2A)^2 + (r^2 - 1) d\Sigma_{2n}^2 + \frac{m-1}{\lambda} r^2 d\Omega_m^2, \quad (37)$$

which is Ricci flat, where

$$\tilde{P}(r) = \lambda r^{1-m} \int_{r_1}^r dx x^{m-2} (x^2 - 1)^n \quad (38)$$

$$= \frac{(-1)^n \lambda}{m-1} \left[{}_2F_1\left(-n, \frac{m-1}{2}; \frac{m+1}{2}; r^2\right) - \left(\frac{r_1}{r}\right)^{m-1} {}_2F_1\left(-n, \frac{m-1}{2}; \frac{m+1}{2}; r_1^2\right) \right]. \quad (39)$$

The radial coordinate runs for $r = r_1$ to $r = \infty$. There are two cases to consider, depending upon whether $r_1 = 1$ or $r_1 > 1$.

5.1 $r_1 > 1$: Generalised Taub-BOLT metrics

Let us first consider the case $r_1 > 1$. Since $\Lambda = 0$ we have from (9) that $\nu = 0$ and hence, analogously to (15), regularity at $r_1 = 1$ implies

$$\frac{k}{p} = \frac{1}{r_1}. \quad (40)$$

Again, therefore, we have that regularity requires $0 < k < p$, as in (18). Thus for each allowed value of k we have a complete, nonsingular, non-compact Ricci-flat metric with $r_1 = p/k$. Topologically, the manifold is S^m times an \mathbb{R}^2 bundle over the Einstein-Kähler base space K_{2n} . At large distance, we have

$$\tilde{P}(r) \sim \frac{\lambda}{2n + m - 1} r^{2n}, \quad (41)$$

and so the coefficient of $(d\tau - 2A)^2$ in (37) becomes asymptotically constant. This is the same type of asymptotic behaviour that is encountered in the Taub-NUT and Taub-BOLT metrics. In fact, the metrics with $r_1 > 1$ that we have obtained here are generalisations of the standard Taub-BOLT metrics obtained in [9, 10]. The new feature is the added S^m factor.

It should be remarked that since the coefficient of $d\Omega_m^2$ remains non-zero in the entire range of the radial variable, we can replace the sphere S^m by any other Einstein space, normalised such that $R_{ij} = (m-1)\delta_{ij}$.

Some explicit low-dimensional examples are as follows. We set $\lambda = 1$ for convenience. For $n = 1$, with $m > 1$, we have

$$\begin{aligned} d\hat{s}_{m+4}^2 &= \frac{(m+1)dr^2}{U} + \frac{4U}{m+1} (d\psi + \cos\theta d\phi)^2 + (r^2 - 1)(d\theta^2 + \sin^2\theta d\phi^2) \\ &\quad + (m-1)r^2 d\Omega_m^2, \\ U &= 1 - \frac{2 + 2^{m-1}(3m-5)r^{1-m}}{(m-1)(r^2-1)}. \end{aligned} \quad (42)$$

The case $n = 1$, $m = 2$ is of some interest, giving

$$d\hat{s}_6^2 = \frac{3r(r-1)dr^2}{(r+1)(r-2)} + \frac{4(r+1)(r-2)}{3r(r-1)} (d\psi + \cos\theta d\phi)^2 + (r^2 - 1)(d\theta^2 + \sin^2\theta d\phi^2) + r^2 d\Omega_2^2. \quad (43)$$

We shall return to this in section (5.2).

For $n = 2$, with $m > 1$, we have

$$\begin{aligned} d\hat{s}_{m+6}^2 &= \frac{(m+3)dr^2}{U} + \frac{U}{m+3} (d\tau - 2A)^2 + (r^2 - 1)d\Sigma_4^2 + (m-1)r^2 d\Omega_m, \\ U &= 1 - \frac{1}{(m^2-1)(r^2-1)^2} \left[4(m-1)r^2 - 4(m+1) \right. \\ &\quad \left. + [(m^2-1)r_1^4 - 2(m-1)(m+3)r_1^2 + (m+1)(m+3)](r_1/r)^{m-1} \right], \end{aligned} \quad (44)$$

where r_1 is 3 or $\frac{3}{2}$, corresponding to $k = 1$ or $k = 2$ respectively, and $d\Sigma_4^2$ is the Fubini-Study metric on \mathbb{CP}^2 , scaled so that $\lambda = 1$.

5.2 $r_1 = 1$: Generalised Taub-NUT metrics

Next, we consider the case when $r_1 = 1$. The discussion of regularity at $r = 1$ is different from the previous one, since now the coefficient of $d\Sigma_{2n}^2$ is also going to zero. Regularity is achieved if and only if K_{2n} is taken to be \mathbb{CP}^n with $d\Sigma_{2n}^2$ its standard Fubini-Study metric. The integer k in (13) must be $k = 1$. Setting $r_1 = 1$ in (38), we find that near $r = 1$, $\tilde{P}(r)$ goes as

$$\tilde{P}(r) \sim \frac{2^n \lambda}{n+1} (r-1)^{n+1}. \quad (45)$$

Setting $r-1 = \rho^2$, and taking $\lambda = 2n+2$ for convenience (this gives the canonical “unit” size for \mathbb{CP}^n), we find that near $\rho = 0$ the metric (37) for $m > 1$ takes the regular form

$$d\hat{s}^2 \sim 2(d\rho^2 + \rho^2 d\tilde{\Omega}_{2n+1}^2) + \frac{m-1}{2n+2} d\Omega_m^2. \quad (46)$$

Here, $d\Omega_{2n+1}^2 = (d\tau - 2A)^2 + d\Sigma_{2n}^2$ is the metric on the unit S^{2n+1} , arising as the Hopf fibration over \mathbb{CP}^n . At large distance, on the other hand, the asymptotic form of the metrics

(37) with $r_1 = 1$ is the same as in our previous discussion in section 5.1. Topologically, these new metrics are non-singular on the manifolds $S^m \times \mathbb{R}^{2n+2}$.

The metrics that we obtain in this case with $r_1 = 1$ are generalisations of the Taub-NUT metrics obtained in higher dimensions in [10]. The new feature is again the addition of the S^m factor. As in the generalised Taub-BOLT metrics in section 5.1, since the coefficient of $d\Omega_m^2$ never vanishes for $r \geq 1$, we can replace the sphere S^m by any Einstein space with $R_{ij} = (m-1)\delta_{ij}$.

Finally, we remark that our discussion of Ricci-flat metrics in this section can be straightforwardly extended to give complete, non-singular, non-compact metrics with negative cosmological constant Λ . Since all the formulae can be immediately read off from the results in section 2, we shall not present them explicitly here.

Some explicit examples of the Ricci-flat generalised Taub-NUT metrics are as follows. We set $\lambda = 1$ for convenience. For $n = 1$, $m > 1$, we have

$$\begin{aligned} d\hat{s}_{m+4}^2 &= \frac{(m+1)dr^2}{U} + \frac{4U}{m+1} (d\psi + \cos\theta d\phi)^2 + (r^2 - 1)(d\theta^2 + \sin^2\theta d\phi^2) \\ &\quad + (m-1)r^2 d\Omega_m^2, \\ U &= 1 - \frac{2 - 2r^{1-m}}{(m-1)(r^2 - 1)}. \end{aligned} \quad (47)$$

The case $n = 1$, $m = 2$ is of some interest, giving

$$d\hat{s}_6^2 = \frac{3r(r+1)dr^2}{(r-1)(r+2)} + \frac{4(r-1)(r+2)}{3r(r+1)} (d\psi + \cos\theta d\phi)^2 + (r^2 - 1)(d\theta^2 + \sin^2\theta d\phi^2) + r^2 d\Omega_2^2. \quad (48)$$

Comparing this with the expression for the six-dimensional generalised Taub-BOLT metric (43), we see that the local form of one transforms into the other under $r \rightarrow -r$. Thus the same local metric form (48) extends smoothly onto two different manifolds, a generalised Taub-NUT metric for $1 \leq r \leq \infty$, and a generalised Taub-BOLT metric for $-\infty \leq r \leq -2$. This feature was also observed for the standard six-dimensional Taub-NUT and Taub-BOLT metrics (and for certain Spin(7) holonomy metrics) in [11].

For $n = 2$, $m > 1$ we have

$$\begin{aligned} d\hat{s}_{m+6}^2 &= \frac{(m+3)dr^2}{U} + \frac{U}{m+3} (d\tau - 2A)^2 + (r^2 - 1)d\Sigma_4^2 + (m-1)r^2 d\Omega_m^2, \\ U &= 1 - 4 \frac{(m-1)r^2 - (m+1) + 2r^{1-m}}{(m^2 - 1)(r^2 - 1)^2}, \end{aligned} \quad (49)$$

where $d\Sigma_4^2$ is the Fubini-Study metric on \mathbb{CP}^2 , scaled so that $\lambda = 1$.

6 Conclusions

In this paper, we have presented new local families of solutions of the Einstein equations, in which the metrics (5) and (10) are constructed as warped products of S^m times two-dimensional S^2 or \mathbb{R}^2 bundles over an Einstein-Kähler space K_{2n} . We have studied the circumstances under which these metrics can extend smoothly onto compact or non-compact manifolds. By this means we have obtained new Einstein metrics with positive Ricci tensor that are topologically S^{m+2} bundles over K_{2n} . We have also obtained new Einstein metrics with positive Ricci tensor on spaces that are warped products of S^m times S^2 bundles over K_{2n} , for $m > 1$. In these latter examples, the S^m itself can be replaced by any Einstein space with positive Ricci tensor. In addition, we have obtained two new classes of complete Ricci-flat non-compact metrics. One class can be thought of as generalisations of the higher-dimensional Taub-NUT metrics, on manifolds with the topologies $S^m \times \mathbb{R}^{n+2}$. The other class can be viewed as generalisations of the higher-dimensional Taub-BOLT metrics, on manifolds with the topologies S^m times \mathbb{R}^2 bundles over K_{2n} . In all these new non-compact examples, which require $m > 1$, we can replace the S^m by any other Einstein space with positive Ricci tensor.

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